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# Quantum statistics of the cyclotron resonance infrared detector

D F Walls

School of Science, University of Waikato, Hamilton, New Zealand

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**Abstract.** A quantum statistical analysis of the interaction of a radiation field with a system of electrons in cyclotron motion in a homogeneous magnetic field is presented. With high magnetic fields the interaction offers a possible means for detecting far infrared radiation. The redistribution of electrons in the Landau ladder of states after interaction with the field is calculated for various initial quantum states. The sensitivity of the detector is shown to depend crucially upon the initial quantum state of the electrons.

## 1. Introduction

Recently a novel technique for the detection of far infrared radiation based on free electron cyclotron resonance in a high magnetic field has been proposed by Robinson (1970, 1973). This is a natural extension of the principle employed in the Putley free carrier photoconductor detector (Putley 1963) to free electrons and should find application at the longer wavelength end of the far infrared.

The operation of the detector can be described in terms of induced transitions between Landau levels with the energy eigenstates of cyclotron motion $W_c = (n+\frac{1}{2})\hbar\omega_c$, $n = 0, 1, 2 \ldots$, where $\omega_c$ is the angular frequency of the orbiting electrons. By a certain configuration of magnetic fields (Robinson 1970a, b, Robinson and Szekeres 1970), electrons are initially pumped into the energy eigenstate $n = mu_0^2/2\hbar\omega_c$ ($m$ is the electron mass and $u_0$ its velocity prior to interaction with the field) or a coherently phased superposition of energy eigenstates (Malkin and Man'ko 1969). The interaction of the cyclotron resonance oscillators with a radiation field then redistributes the electrons within the Landau levels of states. The distribution of electrons is measured by detecting the electrons which overcome an applied bias potential barrier, thus giving a measure of the perturbing field.

A quantum mechanical description of the interaction between the radiation field and the system of electrons undergoing cyclotron motion was advanced by Robinson and Whitbourn (1972). Following Landau and Lifshitz (1965) they represent the Hamiltonian operator for an electron in a homogeneous magnetic field by a quantized harmonic oscillator. However, they treat the radiation field classically. The resulting interaction assumes the form of the forced harmonic oscillator which they treat by the method of Carruthers and Nieto (1965). This is in fact only a semiclassical treatment since a fully quantum mechanical analysis requires the radiation field to be quantized too.

Furthermore, a distinction is not clearly drawn between the different possible initial quantum states for the cyclotron oscillators. Fundamentally different effects

occur depending on whether the cyclotron oscillator is initially in an energy eigenstate (ie a number state $|n\rangle$) or in a distribution of energy eigenstates with a fixed phase (ie a coherent state $|\alpha\rangle$). The solutions for the electron distribution after interaction with the field for a time $t$ are found for the electrons initially in (*a*) a number state $|n\rangle$; (*b*) a coherent state $|\alpha\rangle$. These solutions are compared with the corresponding results obtained classically and semiclassically. Such solutions are familiar from isomorphic problems in quantum optics. In quantum optics solutions for an initial number state of the electromagnetic field are usually of little more than academic interest since a number state is rarely a realistic description of a photon state at optical frequencies. However, since it is possible to achieve a monoenergetic state of orbiting electrons in cyclotron resonance corresponding to a number state, the solutions assume a physical significance. Thus the cyclotron resonance detector offers a possible experimental test of these unverified theoretical results.

## 2. Interaction with a single-mode radiation field

We first consider the interaction of electrons undergoing cyclotron motion with a single mode of the radiation field. Following Landau and Lifshitz (1965) we represent the Hamiltonian operator for an electron in a homogeneous magnetic field by a quantized harmonic oscillator $a$. The single mode of the radiation field is also quantized and represented by the boson annihilation operator $b$. The interaction may be described by the phenomenological Hamiltonian

$$H = \hbar\omega a^\dagger a + \hbar\omega b^\dagger b + \hbar\kappa(a^\dagger b + ab^\dagger) \tag{2.1}$$

where $\kappa$ is the coupling constant and we have assumed resonance. We have dropped the counter rotating terms $a^\dagger b^\dagger + ab$ since they are so far from resonance that a negligible transfer of power occurs (Robinson 1970a).

The Hamiltonian (2.1) is familiar in nonlinear optics as describing the process of parametric frequency conversion. A quantum statistical analysis of this process has been given by Tucker and Walls (1969, to be referred to as I) and Wu (1973). We shall draw on the results of I for our present discussion.

The time-dependent behaviour of the interacting system of radiation field plus electrons is described by the Heisenberg operators (I, equation (2.4)):

$$\begin{aligned} a(t) &= e^{-i\omega t}(a\cos\kappa t - ib\sin\kappa t) \\ b(t) &= e^{-i\omega t}(b\cos\kappa t - ia\sin\kappa t). \end{aligned} \tag{2.2}$$

The statistical properties of the system at time $t$ for any given initial state may be derived using the above operator solutions.

We consider the radiation field to be initially in a chaotic state. This is the state characteristic of, for example, a thermal field. The initial density operator for the field may be expressed in terms of a $P$ representation (Glauber 1963):

$$\rho_b = \int P(\beta)|\beta\rangle\langle\beta|\, d^2\beta \tag{2.3}$$

where $|\beta\rangle$ the coherent states are eigenstates of the annihilation operator $b$. For a

chaotic field the $P$ representation has the form

$$P(\beta) = \frac{1}{\pi\bar{n}} \exp\left(\frac{-|\beta|^2}{\bar{n}}\right) \tag{2.4}$$

where $\bar{n}$ is the mean number of chaotic photons in the field mode.

We wish to consider two possible initial states for the cyclotron oscillators.

### 2.1. *Initial number state* $|n\rangle$

We first consider the case where we have achieved a perfectly monoenergetic system of orbiting electrons, that is a pure energy eigenstate or a number state $|n\rangle$. Our objective is to calculate how the electrons are redistributed in the Landau level of states after interaction with the field for a time $t$. This may best be accomplished by first calculating the $P$ representation for the cyclotron oscillators at time $t$.

This calculation is described in detail in I and here we merely quote the result (I, equation (7.15)) (see also Wu 1973):

$$P(\alpha, t) = \frac{1}{\pi\bar{n}\sin^2\kappa t} \exp\left(\frac{-|\alpha|^2}{\bar{n}\sin^2\kappa t}\right) \left(\frac{\bar{n}\sin^2\kappa t - \cos^2\kappa t}{\bar{n}\sin^2\kappa t}\right)^n L_n\left(\frac{|\alpha|^2\cos^2\kappa t}{\bar{n}\sin^2\kappa t(\cos^2\kappa t - \bar{n}\sin^2\kappa t)}\right) \tag{2.5}$$

where $L_n$ is the Laguerre polynomial. Some care must be taken in the use of this solution since we note that as $\sin^2\kappa t$ approaches zero $P(\alpha, t)$ is a rapidly oscillating function of $\alpha$. At these times the cyclotron oscillators are in the original number state for which the $P$ representation is highly singular. For $\bar{n} \neq 0$, since $L_n(y) > 0$ for $y \leqslant 0$ it follows from equation (2.5) that $P(\alpha, t)$ is positive definite for times such that $\sin^2\kappa t > (1+\bar{n})^{-1}$.

From the $P$ representation we may calculate the distribution $P_m(t)$ in the Landau ladder of eigenstates via the integral

$$P_m(t) = \int P(\alpha, t) \frac{|\alpha|^{2m}}{m!} e^{-|\alpha|^2} d^2\alpha. \tag{2.6}$$

Evaluating this integral (Erdelyi 1954) we arrive at the result

$$\begin{aligned} P_m(t) = {} & (2m!\bar{n}\sin^2\kappa t)^{-1} \left(\frac{\bar{n}\sin^2\kappa t - \cos^2\kappa t}{\bar{n}\sin^2\kappa t}\right)^n \\ & \times \left(\frac{\bar{n}\sin^2\kappa t(\cos^2\kappa t - \bar{n}\sin^2\kappa t)}{\cos^2\kappa t}\right)^{m+1} \frac{\Gamma(m+n+1)}{n!} \frac{(\psi-1)^n}{\psi^{m+n+1}} \\ & \times {}_2F_1\left(-n, -m, -m-n, \frac{\psi}{\psi-1}\right) \end{aligned} \tag{2.7}$$

where

$$\psi = \left(1 + \frac{1}{\bar{n}\sin^2\kappa t}\right)\left(\frac{\bar{n}\sin^2\kappa t(\cos^2\kappa t - \bar{n}\sin^2\kappa t)}{\cos^2\kappa t}\right),$$

$\Gamma$ is the gamma function and ${}_2F_1$ the hypergeometric function.

We note a difference in the expression (2.7) to that obtained by the semiclassical analysis of Robinson and Whitbourn. The latter analysis, however, suffers from the inability to include the statistical state of the radiation field.

The experimental measurement of the distribution of electrons after interaction by means of a variable bias potential barrier is described by Robinson (1970a, 1973).

The total energy absorbed from the radiation field by the electrons may be calculated in the Heisenberg picture as follows.

The energy absorbed after a time $t$ is given by

$$\Delta E = E(t) - E(0) = \mathrm{Tr}(\rho(0)H_a(t)) - \mathrm{Tr}(\rho(0)H_a(0)) \tag{2.8}$$

where $\rho(0)$ is the initial density operator for the system of electrons plus radiation field:

$$\rho(0) = (|n\rangle\langle n|)_a \frac{1}{\pi\bar{n}} \int \mathrm{e}^{-|\beta|^2/\bar{n}} |\beta\rangle\langle\beta| \, \mathrm{d}^2\beta. \tag{2.9}$$

The Hamiltonian $H_a(t)$ for the cyclotron oscillators may be written using the solutions given by equation (2.2) as

$$\begin{aligned} H_a(t) &= \hbar\omega a^\dagger(t)a(t) \\ &= \hbar\omega(a^\dagger(0)a(0)\cos^2\kappa t + b^\dagger(0)b(0)\sin^2\kappa t + \mathrm{Im}\, a(0)b^\dagger(0)\sin 2\kappa t). \end{aligned} \tag{2.10}$$

Evaluation of the trace in equation (2.8) yields

$$\Delta E = \hbar\omega(\bar{n} - n)\sin^2\kappa t. \tag{2.11}$$

Thus a nett absorption of radiation results provided $\bar{n} > n$. This expression corresponds to the phase averaged term $W_1$ derived in the classical treatment (Robinson 1970).

The third term in equation (2.10) is phase-dependent and gives a zero contribution when the oscillator is in a number state. This is because the number state has a completely unspecified phase. This corresponds to the classical situation where averaging over completely random phases yields a null result. As we shall see in the next subsection the sensitivity of the detector is enhanced when the electrons are in a state of fixed phase.

### *2.2. Initial coherent state*

Instead of preparing the electrons in a pure energy eigenstate one may allow the electrons to occupy a distribution of energy eigenstates such that one has a state with a definite phase. That is, one prepares the cyclotron oscillators in a coherent state $|\alpha_0\rangle$. These coherent states have been studied extensively in the context of electromagnetic radiation (see for example Glauber 1963). Malkin and Man'ko (1969) have shown how they may be useful in describing the motion of a charged particle in a magnetic field. For a coherent state the initial distribution in the Landau level of states is Poissonian:

$$P_n(0) = |\alpha_0|^{2n} \mathrm{e}^{-|\alpha_0|^2}/n! \tag{2.12}$$

with mean $|\alpha_0|^2$.

After interaction with a chaotic radiation field for a time $t$, the $P$ representation for the cyclotron oscillators is given by (I, equation (6.3))

$$P(\alpha, t) = \frac{1}{\pi\bar{m}\sin^2\kappa t} \exp\left(-\frac{|\alpha - \bar{\alpha}(t)|^2}{\bar{m}\sin^2\kappa t}\right) \tag{2.13}$$

where

$$\bar{\alpha}(t) = e^{-i\omega t}\alpha_0 \cos \kappa t.$$

Thus the cyclotron oscillators are no longer in a coherent state but a mixture of coherent and chaotic states, the noisy character having been introduced through the interaction with the thermal field.

The distribution of the electrons in the Landau ladder of states may be obtained via the integral equation (2.6). Evaluation of this integral (Lachs 1965, Glauber 1966) yields

$$P(n) = \frac{1}{1+\bar{m}}\left(\frac{\bar{m}}{1+\bar{m}}\right)^n \exp\left(\frac{-|\alpha_0|^2 \cos^2 \kappa t}{\bar{m}+1}\right) L_n\left(\frac{-|\alpha_0|^2 \cos^2 \kappa t}{\bar{m}(\bar{m}+1)}\right). \tag{2.14}$$

The total energy absorbed from the radiation field after interaction for a long time $t$ may be calculated from equation (2.8), where now

$$\rho(0) = |\alpha_0\rangle\langle\alpha_0| \frac{1}{\pi\bar{n}} \int e^{-|\beta|^2/\bar{n}} |\beta\rangle\langle\beta| \, d^2\beta. \tag{2.15}$$

Evaluation of the trace yields

$$\Delta E = \hbar\omega(\bar{n}-n) \sin^2 \kappa t + |\alpha_0| \bar{n}^{1/2} \sin \phi \sin 2\kappa t \tag{2.16}$$

where $\alpha_0 = |\alpha_0| e^{i\phi}$.

This corresponds to the expression $\Delta E = W_1 + W_2$ obtained classically (Robinson 1970). The second term corresponding to $W_2$ arises from the third term in equation (2.10). This phase-dependent term yields a finite contribution for a coherent state since the coherent state has a fixed phase. The correspondence of the classical system of electrons with a fixed phase with the coherent states of quantum mechanics is clear.

The additional energy absorbed resulting from the term $W_2$ is, as Robinson notes, relatively large and should be easily measurable. Robinson has shown that when a system of electrons with 20 eV initial energy is brought into resonant interaction for a time $t = 10^{-7}$ s with a field $E_0 = 5\,\mathrm{V m^{-1}}$, the mean energy acquired is 0·022 eV while the maximum energy $\Delta E$ reaches 1·522 eV.

This distinction between the possible initial quantum states of the cyclotron oscillators and the considerable difference in the results obtained therefrom is not made clear in Robinson and Whitbourn's analysis. They calculate the redistribution of electrons in the Landau ladder starting with an initial number state; they then show that the energy transfer for an initial coherent cyclotron oscillation state is given by $W_1 + W_2$ without clearly elucidating the fundamental difference between these two quantum states.

From our analysis it is clear that to optimize the detector efficiency one requires the electrons to be in a coherent state initially. However, the possibility of preparing initial number states in cyclotron resonance to test formulae applied in quantum optics where experimental testing is not feasible offers another application of the proposed detector.

## 3. Interaction with a broadband radiation field

The assumption of a single-mode radiation field is of course an unrealistic one, since one is normally interested in detecting a broadband radiation field. The interaction of

the cyclotron oscillators with a broadband field may be described by the phenomenological Hamiltonian

$$H = \hbar\omega_a a^\dagger a + \hbar \sum \omega_j b_j^\dagger b_j + \hbar \left( a^\dagger \sum \kappa_j b_j + a \sum \kappa_j^* b_j^\dagger \right) \tag{3.1}$$

where the $b_j$ are the annihilation operators for the field modes with frequency $\omega_j$.

The probability distribution of electrons in the occupied Landau levels is then best obtained using master equation techniques. In fact the above Hamiltonian describes the interaction of a single harmonic oscillator with a large reservoir of harmonic oscillators.

A master equation for $P_m(t)$, the probability of the $m$th level being occupied at time $t$, may be derived under the assumption that the density operator of the radiation field is not appreciably altered during the interaction. That is, the radiation field is of sufficient strength so as to be relatively undepleted during the interaction. The radiation field is assumed to be a thermal field with density operator

$$\rho_b(0) = \exp(-\beta \Sigma \hbar\omega_j b_j^\dagger b_j)/\mathrm{Tr}(\exp(-\beta \Sigma \hbar\omega_j b_j^\dagger b_j)) \tag{3.2}$$

where $\beta = 1/kT$.

The resultant Markoffian master equation is well known (see the review article by Agarwal (1973) and references contained therein) and assumes the following form in the Fock representation:

$$\frac{\mathrm{d}P_m}{\mathrm{d}t} = -\gamma\bar{n}(m+1)P_m + \gamma(\bar{n}+1)(m+1)P_{m+1} - \gamma(\bar{n}+1)mP_m + \gamma\bar{n}mP_{m-1} \tag{3.3}$$

where $\bar{n}$ is the mean number of photons per mode in the radiation field evaluated about $\omega_a$ and $\gamma = 2\pi g(\omega_a)|\kappa(\omega_a)|^2$, where $g(\omega_j)$ is the density of the radiation field modes. Again we consider solutions to this equation for two particular initial states of the cyclotron oscillators.

### *3.1. Initial number state*

For the case where one has prepared the orbiting electrons in an energy eigenstate $|n\rangle$ we must solve equation (3.3) subject to the initial condition

$$P_m(0) = \delta_{nm}. \tag{3.4}$$

A solution to (3.3) for the initial condition (3.4) has been given by Agarwal (1969) and Schell and Barakat (1973). The result is

$$P_m(t) = \frac{a^m}{(1+a)^{m+1}} \left( \frac{a\,\mathrm{e}^{\gamma t} - 1}{a\,\mathrm{e}^{\gamma t}} \right)^n {}_2F_1(-n, m+1, 1, \lambda^{-1}) \tag{3.5}$$

where

$$a = \bar{n}(1 - \mathrm{e}^{-\gamma t})$$

$$\lambda = (1 - a\,\mathrm{e}^{\gamma t})(1+a).$$

Again this result differs from the semiclassical expression (Robinson and Whitbourn 1973), since in our formalism we are able to include realistic information as to the spectral nature and statistical properties of the radiation field. The distribution assumes a similar functional form to the expression (2.7) obtained for the single-mode case (allowing for the difference in time-dependent behaviour).

*3.2. Initial coherent state*

If the electrons are initially prepared in a phased coherent state the initial density operator for the cyclotron oscillators is

$$\rho(0) = |\alpha_0\rangle\langle\alpha_0|. \tag{3.6}$$

To obtain the electron distribution at time $t$ it is more convenient to work with the Fokker–Planck equation corresponding to equation (3.3). The Fokker–Planck equation for the $P$ function for the cyclotron oscillators is given by

$$\frac{\partial P}{\partial t} = \frac{\gamma}{2}\left(\frac{\partial}{\partial\alpha}(\alpha P) + \frac{\partial}{\partial\alpha^*}(\alpha^* P)\right) + \gamma\bar{n}\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}. \tag{3.7}$$

The solution to this equation for the initial coherent state equation (3.6) corresponding to the initial condition $P(\alpha, 0) = \delta^2(\alpha - \alpha_0)$ is

$$P(\alpha, t) = [\pi\bar{n}(1 - e^{-\gamma t})]^{-1} \exp\left(\frac{-|\alpha - \alpha_0 e^{-\gamma t/2}|^2}{\bar{n}(1 - e^{-\gamma t})}\right). \tag{3.8}$$

Again we note the noisy character of the distribution resulting from the interaction with the thermal field. The distribution of electrons in the Landau levels is obtained from equation (3.8) via the integral equation (2.6) with the result

$$P_n(t) = \frac{1}{(1+\bar{n})}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n \exp\left(\frac{-|\alpha_0|^2 e^{-\gamma t}}{\bar{n}(t)+1}\right) L_n\left(\frac{-|\alpha_0|^2 e^{-\gamma t}}{\bar{n}(t)(\bar{n}(t)+1)}\right) \tag{3.9}$$

where $\bar{n}(t) = \bar{n}(1 - e^{-\gamma t})$.

One notes that the same functional form results as for the single-mode radiation field equation (2.14), apart from a difference in the time-dependent behaviour. For short times $\gamma t, \kappa t \ll 1$, the time-dependent behaviour also becomes identical.

## 4. Summary

The quantum statistical properties of the cyclotron resonance detector have been discussed based on appropriate model Hamiltonians. The response of the detector is strongly dependent upon the initial quantum state of the cyclotron oscillators. The average energy absorbed from the radiation field when the cyclotron oscillators are initially in an $n$ quantum state is considerably less than when the initial state is a coherent superposition of number states. This difference stems from the intrinsic uncertainty in phase possessed by a number state, thus yielding a null contribution when tracing over the phase-dependent term of the Hamiltonian.

Thus for maximum sensitivity the cyclotron oscillators should initially be prepared in a coherent state. However, the possibility of preparing an initial $n$ quantum state offers the intriguing possibility of testing theory commonly used in quantum optics, where the preparation of an $n$ photon state is not feasible.

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